SNC log symplectic structures on Fano products

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Introduction to Poisson structures

Definition (Poisson bracket).

The holomorphic **Poisson bracket** on X:

- (bilinear form) $\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$
- (skew-symmetric) $\{f,g\} = -\{g,f\}$,
- (Jacobi identity) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$,
- (Leibniz rule) $\{f, g\cdot h\} = \{f,g\}h + \{f,h\}g$.

Definition (Poisson structure).

The holomorphic **Poisson structure** on the smooth variety X:
$\Pi \in \Gamma(X, \wedge^2\mathcal{T}_X)$ s.t. $[\Pi,\Pi] = 0 \in \wedge^3\mathcal{T}_X$, where $[-,-]$ is the Schouten bracket.
$[f\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}, g\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial w}] := f\frac{\partial g}{\partial x}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial w} + f\frac{\partial g}{\partial y}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial w} + g\frac{\partial f}{\partial z}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial w} + g\frac{\partial f}{\partial w}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$

Remark.

- $\{$Poisson structures on $X\} \leftrightarrow \{$Poisson brackets on $X\}$

$(\because)$
$\to : \Pi : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X; (f,g) \mapsto \Pi^\#(df, dg)$
$\leftarrow : \Pi = \Sigma_{i,j}\{x_i, x_j\}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$

- $[\Pi,\Pi] = 0 \iff$ Jacobi identity

Definition (Degeneracy divisor).

- Poisson structure (X,Π) is **generically symplectic**
$\iff$ $\dim X = 2n$ and $\Pi^n \neq 0$.

Suppose that (X,Π) is a generically symplectic.

- $D(\Pi) := \{x \in X \mid \Pi^n(x) = 0\}$ forms a divisor called the **degeneracy divisor**,
- (X,Π) is a **log symplectic strucuture**
$\iff$ $D(\Pi)$ is a reduced divisor,
- (X,Π) is a **SNC log symplectic structure**
$\iff$ (X,Π) is a log symplectic structure and $D(\Pi)$ is a simple normal crossing divisor.

$\Pi^n \in \Gamma(X, \wedge^{2n}\mathcal{T}_X) \rightsquigarrow D(\Pi) \sim -\mathcal{K}_X$.

Motivations & Main Result

Main Theorem

X_i: Fano variety over $\mathbb{C}$, $\mathrm{Pic}(X_i) = \mathbb{Z}$, $\dim X_i = n_i \geq 3$,
$X = \Pi_{i=1}^m X_i$, $\dim X = 2n$,
Π: SNC log symplectic structure.

$\Rightarrow$
- $X_i = \mathbb{P}^{n_i}$
- Π : diagonal Poisson structure

Background

Question

How many (X,Π) with conditions
- X : smooth projective variety
- $D(\Pi)$: reduced SNC

Theorem (Lima, Pereira).

X: Fano variety over $\mathbb{C}$, $\mathrm{Pic}(X) = \mathbb{Z}$, $\dim X = 2n \geq 4$,
Π: SNC log symplectic structure on X.

$\Rightarrow$
- $X = \mathbb{P}^n$
- Π : diagonal Poisson structure on $\mathbb{P}^n$

How about if $\rho(X) \geq 2$

Corollary (O).

X: Fano variety over $\mathbb{C}$, $\mathrm{Pic}(X) = \mathbb{Z}$,
$\dim X \geq 3$,
$X = \mathbb{P}^n \Leftrightarrow \exists\Pi$:SNC log symplectic structure on $X \times X$

Diagonal Poisson structures and form as bivector fields

Definition (Diagonal Poisson structure).

X :$\mathbb{A}^{2n}$ or (product of) $\mathbb{P}^n$, $\dim X = 2n$
(X,Π) is a **diagonal Poisson structure**
$\iff$ $D(\Pi)$ is composed of all coordinate hyperplanes.

Theorem (Polishchuk).

There is a surjective map of bivector fields:

$$\varphi_n : \{(\mathbb{A}^{n+1}, \bar{\Pi}) \mid \bar{\Pi} : \text{quadratic}\} \to \{(\mathbb{P}^n, \Pi)\}$$

Furthermore, $\bar{\Pi}$ is Poisson on $\mathbb{A}^{n+1} \Rightarrow \varphi_n(\bar{\Pi})$ is also Poisson on $\mathbb{P}^n$.

$(\because)$

$$\{\frac{x_i}{x_k}, \frac{x_j}{x_k}\} = \frac{1}{{x_k}^2}\{x_i, x_j\} - \frac{x_i}{{x_k}^3}\{x_k, x_j\} - \frac{x_j}{{x_k}^3}\{x_i, x_k\}$$

$\{x_i, x_j\}$ is quadratic, $\rightsquigarrow \{\frac{x_i}{x_k}, \frac{x_j}{x_k}\} \in k[\frac{x_0}{x_k}, \cdots, \frac{x_n}{x_k}]$ $\square$

Fact (Pym).

$X := \prod_{i=1}^m \mathbb{P}^{n_i}$, $\dim X = \sum_{i=1}^m n_i = 2n$,$\Pi$: diagonal Poisson str. on X,
coordinate:$[x_{10}, \cdots : x_{1n_1} : x_{20} : \cdots : x_{2n_2} : \cdots : x_{mn_m}]$,
$\Rightarrow \exists\sigma = \sum_{1\leq i,k\leq m, 1\leq j\leq n_i, 1\leq m\leq n_k} \Delta_{ijkl}$: diagonal Poisson str. on $\mathbb{A}^{2n-m} \simeq \prod_{i=1}^m \mathbb{A}^{n_i-1}$, where $\Delta_{ij} = c_{ijkl}x_{ij}x_{kl}\frac{\partial}{\partial x_{ij}}\wedge\frac{\partial}{\partial x_{kl}}$
s.t. σ induces Π on X.

$(\because)$

Definition. (r-matrix construction)

Π is constructed by **r-matrix construction** w.r.t a Lie group G:
Π is a image of r along $\mathfrak{g} \to \Gamma(X, \mathcal{T}_X)$,
where $\mathfrak{g}$: Lie algebra of G, r: r-matrix for G, i.e. $[r,r] = 0$.

The Fact comes from r-matrix construction for
$G = (\mathbb{C}^*)^{n_1} \times (\mathbb{C}^*)^{n_2} \times \cdots \times (\mathbb{C}^*)^{n_m}$. $\square$

Key lemma (Pym).

(X,Π): SNC log symplectic structure
$D(\Pi) = \Sigma_{j=1}^k D_j$: irreducible decomposition of the degeneracy divisor.
$\Rightarrow \mathrm{ch}(\mathcal{T}_X) - \mathrm{ch}(\mathcal{T}_X^\vee) = 2\sinh[D_j]$

$(\because)$ We have 2 exact sequences;

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \to \bigoplus_{j=1}^m \mathcal{O}_{D_j} \to 0,$$

$$0 \to \mathcal{O}_X(-D_j) \to \mathcal{O}_X \to \mathcal{O}_{D_j} \to 0.$$